Solving Integrals with the Quantum Computer Algebra System

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Quantum is a small computer algebra system which includes new implementations of our ISSAC 96 [7] and ISSAC 97 [8] algorithms for computing hypergeometric function representations and Meijer G function representations, as well as a repertoire of essential computer algebra system algorithms necessary for solving integral calculus problems.

On Apr 18 2005, as demonstration of the power of our methods, we published 100,000+ hypergeometric formulas on our PLANETQUANTUM.COM web site. We created these formulas using Quantum. The algorithms and techniques used included our ISSAC 96 and ISSAC 97 algorithms as well as symbolic linear algebra, multivariate polynomial factorization, multivariate polynomial gcd, partial fraction decomposition, high precision numerical evaluation, special functions [1], integration, differentiation, integer factorization, the LLL algorithm, Simon Plouffe et. al.'s Inverse Symbolic Calculator, Neil Sloane's On-Line Encyclopedia of Integer Sequences, and many books and references including Prudnikov, A. P., Brychkov, Yu. A., Marichev, O. I., [5].

Since our successful publication of 100,000+ hypergeometric formulas, we have tasked ourselves with the new challenging goal of online publication of 100,000+ integrals at the same difficulty level as integrals which can be found in famous integral tables such as Gradshteyn, I. S. and Ryzhik, I. M. [2] and Prudnikov, A. P., Brychkov, Yu. A., Marichev, O. I., [3], [4], [5]. Our attention has focussed back again on the Risch algorithm [6], Meijer G theory, hypergeometric integrals, and most recently, elliptic integrals. Example integrals of this nature would be:

$$
\begin{aligned}
&\int_{-\infty}^{-3} \frac{\left(\left(z^2+1\right)\left(z^2-4\right)\right)^{-1/2}}{z-5}\,dz \\
&= \frac{1}{5^{3/2}}\,\mathrm{F}\left(\frac{\sqrt{5}}{3}, \frac{1}{\sqrt{5}}\right) - \frac{1}{5^{3/2}}\,\mathrm{K}\left(\frac{1}{\sqrt{5}}\right) + \frac{1}{\sqrt{546}}\tanh^{-1}\left(\sqrt{\frac{21}{26}}\right) \\
&\quad - \frac{1}{\sqrt{546}}\tanh^{-1}\left(\sqrt{\frac{13}{21}}\right) + \frac{4}{21\cdot 5^{3/2}}\,\Pi\left(\frac{\sqrt{5}}{3}, \frac{25}{21}, \frac{1}{\sqrt{5}}\right) - \frac{4}{21\cdot 5^{3/2}}\,\Pi\left(\frac{25}{21}, \frac{1}{\sqrt{5}}\right)
\end{aligned}
$$

and

$$
\int_0^\infty e^{-\alpha\,x}\,J_0\left(\beta\,x\right)^2\,dx = \frac{2}{\pi\,\sqrt{\alpha^2+4\,\beta^2}}\,\mathrm{K}\left(\frac{2\,\beta}{\sqrt{\alpha^2+4\,\beta^2}}\right)
$$

References

[1] Abramowitz, M. and Stegun I. A. (eds.) (1965), *Handbook of Mathematical Functions*, Dover Publications, Inc., New York.

[2] Gradshteyn, I. S. and Ryzhik, I. M. (2007), *Table of Integrals, Series, and Products*, Academic Press.

[3] Prudnikov, A. P., Brychkov, Yu. A., Marichev O. I. (1986), *Integrals and Series, Volume 1: Elementary Functions*, Gordon and Breach Science Publishers.

[4] Prudnikov, A. P., Brychkov, Yu. A., Marichev O. I. (1986), *Integrals and Series, Volume 2: Special Functions*, Gordon and Breach Science Publishers.

[5] Prudnikov, A. P., Brychkov, Yu. A., Marichev O. I. (1990), *Integrals and Series, Volume 3: More Special Functions*, Gordon and Breach Science Publishers.

[6] Risch, R. H. (1969), "The Problem of Integration in Finite Terms", *Transactions of the American Mathematical Society*, **139**, 167–189.

[7] Roach, K. (1996), "Hypergeometric Function Representations", *Proceedings of ISSAC '96*, 301–308. ACM, New York.

[8] Roach, K. (1997), "Meijer G Function Representations", *Proceedings of ISSAC '97*, 205–211. ACM, New York.

$\int_0^a \left(a^2-x^2\right)^{n-1/2}\, dx = \Gamma\left[\begin{matrix}\frac{2\,n+1}{2}\\ n+1\end{matrix}\right]\,\frac{\sqrt{\pi}\,a^{2\,n}}{2}$
$\int_0^\infty e^{-b\,x-x^2/4\,a}\, dx = \sqrt{a\,\pi}\,e^{a\,b^2}\,\mathrm{erfc}\left(b\,\sqrt{a}\right)$
$\int_0^\infty \frac{x}{\left(e^{3\,x}-1\right)^{1/3}}\, dx = \frac{\pi\log(3)}{3^{3/2}}+\frac{\pi^2}{27}$
$\int_0^\infty \frac{x\,e^{-x\,n}}{e^x+1}\, dx = \Phi\left(-1,2,n+1\right)$
$\int_0^1 \sin^{-1}(x)\,x^{2\,n-1}\, dx = \frac{\pi}{4\,n}-\frac{\sqrt{\pi}}{4\,n}\,\Gamma\left[\begin{matrix}\frac{2\,n+1}{2}\\ n+1\end{matrix}\right]$
$\int_0^\infty \frac{x^{a-1}}{\sqrt{x^b+1}}\, dx = \frac{1}{b\,\sqrt{\pi}}\,\Gamma\left[\frac{b-2\,a}{2\,b},\frac{a}{b}\right]$
$\int_0^{\frac{\pi}{4}} \cos(x)^{-2\,a-2}\cos\left(2\,x\right)^a\, dx = \frac{\sqrt{\pi}}{2}\,\Gamma\left[\begin{matrix}a+1\\ \frac{2\,a+3}{2}\end{matrix}\right]$
$\int_0^\infty \frac{x\,e^{-2\,x\,n}}{e^x+1}\, dx = \Phi\left(-1,2,2\,n+1\right)$
$\int_0^\infty \frac{x\,e^{-3\,x}}{e^{-x}+1}\, dx = \frac{\pi^2}{12}-\frac{3}{4}$
$\int_0^\infty x^{b-1}\,\left(a\,x+1\right)^{-c}\, dx = a^{-b}\,\Gamma\left[\begin{matrix}b,c-b\\ c\end{matrix}\right]$
$\int_0^\infty \left(\mathrm{Si}\left(a\,x\right)-\frac{\pi}{2}\right)\,e^{-b\,x}\, dx = \frac{1}{b}\tan^{-1}\left(\frac{a}{b}\right)-\frac{\pi}{2\,b}$

$$\int_0^\infty \frac{x\, e^{-x}}{\sqrt{e^x-1}}\, dx = \pi \log(2) - \frac{\pi}{2}$$
$$\int_0^\infty x^{a-1}\, (b+x)^{-n}\, dx = \Gamma\left[\begin{matrix} a, n-a \\ n \end{matrix}\right]\, b^{a-n}$$
$$\int_0^\infty J_1\left(a\, x\right)\, e^{-x^2\, b}\, dx = \frac{1}{a} - \left(e^{-a^2}\, a^{-4\, b}\right)^{1/4\, b}$$
$$\int_0^\infty \frac{e^{-a\, x}\, x^n}{b+x}\, dx = e^{a\, b}\, \Gamma\left(n+1\right)\, b^n\, \Gamma\left(-n, a\, b\right)$$
$$\int_0^\infty \frac{\cot^{-1}(x)}{x+1}\, dx = G + \frac{\pi \log(2)}{4}$$
$$\int_0^\infty \frac{\log\left(x+1\right)\, \cos\left(a\, x\right)}{x}\, dx = \frac{\mathrm{Ci}(a)^2}{2} + \frac{\mathrm{Si}(a)^2}{2} - \frac{\pi\, \mathrm{Si}(a)}{2} + \frac{\pi^2}{8}$$
$$\int_0^\infty \frac{\tan^{-1}(x)}{1-x^2}\, dx = -G$$
$$\int_0^\infty \frac{J_\mu\left(a\, x\right)}{b+x}\, dx = \pi\, \mathbf{J}_\mu\left(a\, b\right)\, \csc\left(\mu\, \pi\right) - \pi\, J_\mu\left(a\, b\right)\, \csc\left(\mu\, \pi\right)$$
$$\int_0^\infty x^2\, \mathrm{csch}(x)\, e^{-2\, x\, n}\, dx = \frac{1}{2}\, \zeta\left(3, \frac{2\, n+1}{2}\right)$$
$$\int_0^\infty x\, \mathrm{erfc}\left(a\, x\right)\, e^{-x^2\, b}\, dx = \frac{1}{2\, b} - \frac{a}{2\, b\, \sqrt{a^2+b}}$$
$$\int_0^\infty \mathrm{Ei}\left(-x^2\right)\, e^{x^2\, a^2}\, dx = \frac{\pi^{3/2}}{2\, a} - \frac{\sqrt{\pi}\, \sin^{-1}(a)}{a}$$

$$\int_{2}^{\infty} \frac{1}{(z+14)\,\sqrt{(z-1)\,\left((z-13)^2+81\right)}}\,dz$$ $$= \frac{1}{15^{3/2}}\,\mathrm{K}\left(\frac{3}{\sqrt{10}}\right) - \frac{1}{2\cdot 15^{3/2}}\,\mathrm{F}\left(\frac{\sqrt{15}}{8}, \frac{3}{\sqrt{10}}\right) - \frac{1}{5\,\sqrt{486}}\,\sin^{-1}\left(\frac{3^{3/2}}{2^{7/2}}\right)$$
$$\int_{-\infty}^{0} \frac{1}{(18-z)\,\sqrt{(1-z)\,\left((16-z)^2+64\right)}}\,dz$$ $$= \frac{1}{17^{3/2}}\,\mathrm{K}\left(\frac{1}{\sqrt{17}}\right) - \frac{1}{2\cdot 17^{3/2}}\,\mathrm{F}\left(\frac{\sqrt{17}}{9}, \frac{1}{\sqrt{17}}\right) - \frac{\sin^{-1}\left(\frac{1}{9}\right)}{34}$$
$$\int_{1}^{2} \sqrt{\frac{z^2-1}{9-z^2}}\,dz = 4\,\mathrm{E}\left(\sqrt{\frac{2}{3}}, \frac{1}{2}\right) - 2\,\mathrm{F}\left(\sqrt{\frac{2}{3}}, \frac{1}{2}\right) - \sqrt{\frac{5}{3}}$$
$$\int_{4}^{5} \frac{1}{(z+5)\,\sqrt{(z-1)\,(5-z)\,\left((z-1)^2+9\right)}}\,dz$$ $$= \frac{2}{15^{3/2}}\,\mathrm{F}\left(\frac{\sqrt{5}}{3}, \frac{1}{\sqrt{5}}\right) - \frac{\sin^{-1}\left(\frac{1}{3}\right)}{10\cdot 3^{3/2}}$$
$$\int_{2}^{3} \sqrt{\frac{z}{(z-1)\,(z-2)}}\,dz$$ $$= \sqrt{6} + 2^{3/2}\,\mathrm{F}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - 2^{3/2}\,\mathrm{E}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$\int_{3}^{4} \frac{1}{(6-z)\,\sqrt{(5-z)\,(z-2)\,(z-3)\,z^3}}\,dz$$ $$= \frac{1}{12}\,\mathrm{F}\left(\frac{\sqrt{3}}{2}, \frac{2}{3}\right) - \frac{1}{30}\,\mathrm{E}\left(\frac{\sqrt{3}}{2}, \frac{2}{3}\right) + \frac{1}{108}\,\Pi\left(\frac{\sqrt{3}}{2}, \frac{8}{9}, \frac{2}{3}\right) + \frac{1}{45\cdot 2^{3/2}}$$

$\mathrm{F}\left(-\frac{26}{7},-\frac{26}{7};1;-1\right)=\frac{95\,\Gamma\left[\frac{1}{7},\frac{5}{14}\right]}{39\cdot 2^{2/7}\,\pi^{3/2}}\cos\left(\frac{3\,\pi}{14}\right)$
$\mathrm{F}\left(-\frac{15}{4},1,1;-\frac{7}{2},2;1\right)=\frac{2252}{665}+\frac{18\log(8)}{19}+\frac{9\,\pi}{19}-\frac{36\log(2)}{19}$
$\mathrm{F}\left(1,1,\frac{3}{2};\frac{1}{2},\frac{9}{2};-1\right)=\frac{1841}{15}-49\cdot 2^{3/2}\log\left(\sqrt{2}+1\right)$
$\mathrm{F}\left(-\frac{1}{2},-\frac{1}{2},\frac{5}{2};\frac{3}{2},\frac{7}{2};-1\right)=\frac{805}{9\cdot 2^{13/2}}-\frac{15\log\left(\sqrt{2}+1\right)}{128}$
$\mathrm{F}\left(-\frac{11}{4},-\frac{5}{2},-\frac{5}{2};-\frac{7}{2},\frac{1}{4};1\right)=\frac{355\,\Gamma\left[\frac{1}{4},\frac{1}{4}\right]}{7\cdot 2^{7/2}\sqrt{\pi}}$
$\mathrm{F}\left(\frac{1}{2},\frac{1}{2};1;\frac{9}{10}\right)=\frac{2}{\pi}\,\mathrm{K}\left(\frac{3}{\sqrt{10}}\right)$
$\mathrm{F}\left(1,1;\frac{2}{3};\frac{8}{9}\right)=3^{3/2}\,\pi+9-3\log(3)$
$\mathrm{F}\left(1,1;\frac{1}{2};-\frac{7}{8}\right)=\frac{8}{15}-\frac{8\sqrt{7}}{15^{3/2}}\sinh^{-1}\left(\frac{\sqrt{7}}{2^{3/2}}\right)$
$\mathrm{F}\left(1,3;2,4,5;-z\right)=\frac{144\,J_1\left(2\sqrt{z}\right)}{z^{7/2}}-\frac{6\,\left(12-z^2\right)}{z^3}-\frac{72\,J_0\left(2\sqrt{z}\right)}{z^3}$
$\mathrm{F}\left(-\frac{7}{2};-\frac{5}{2};-z\right)=\frac{\left(4\,z^2-8\,z^3-6\,z+15\right)\,e^{-z}}{15}-\frac{8\sqrt{\pi}\,z^{7/2}\,\mathrm{erf}\left(\sqrt{z}\right)}{15}$
$\mathrm{F}\left(2;\frac{9}{4};z\right)=\frac{5\,\left(1-4\,z\right)\,\Gamma\left(\frac{5}{4},z\right)\,e^{z}}{16\,z^{5/4}}-\frac{5\,\left(1-4\,z\right)\,\Gamma\left(\frac{1}{4}\right)\,e^{z}}{64\,z^{5/4}}+\frac{5}{4}$