An n-Dimensional Integral Formula

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1. Introduction

This note discusses a generalization of some formulas in "A Heuristic for Exact Calculation of n-Dimensional n+1-Parametric Integrals of Some Elementary and Special Functions" by Vladimir V. Bondarenko. The following simple theorem can be used in combination with the more interesting results appearing in the next two sections.

Theorem. *Suppose* $\alpha \neq 0$ *and* $\int \mathrm{f}(z)\, dz^n = \mathrm{F}(z)$.
Then $\int \mathrm{f}\left(\alpha\, z + \beta\right)\, dz^n = \alpha^{-n}\, \mathrm{F}\left(\alpha\, z + \beta\right)$.

2. An n-Dimensional F Integral

Theorem. *Assume* μ *is not a negative integer and* ρ *is a positive integer. Then*

$$\int z^{\mu}\, \mathrm{F}\left(\vec{a}; \vec{b}; c\, z^{\rho}\right)\, dz^n$$

$$= \Gamma\left(\begin{matrix} \mu + 1 \\ \mu + n + 1 \end{matrix}\right)\, z^{\mu+n}\, \mathrm{F}\left(\Delta\left(\mu + 1, \rho\right), \vec{a}; \Delta\left(\mu + n + 1, \rho\right), \vec{b}; c\, z^{\rho}\right)$$

Comment. Since

$$\mathrm{F}\left(\Delta\left(\mu+1,\rho\right),\vec{a};\Delta\left(\mu+n+1,\rho\right),\vec{b};c\,z^{\rho}\right)$$

for $n \in N$ are generated by a finite dimensional basis over $Q(\mu, \vec{a}, \vec{b}, z)$, the indefinite integrals

$$\begin{aligned} I_n &= \int z^{\mu}\,\mathrm{F}\left(\vec{a};\vec{b};c\,z^{\rho}\right)\,dz^n \\ &= \Gamma\left(\begin{matrix}\mu+1\\ \mu+n+1\end{matrix}\right)\,z^{\mu+n}\,\mathrm{F}\left(\Delta\left(\mu+1,\rho\right),\vec{a};\Delta\left(\mu+n+1,\rho\right),\vec{b};c\,z^{\rho}\right)\end{aligned}$$

satisfy a linear recurrence relation. This linear recurrence relation can be explicitly computed.

Example.

$$I_n = \int \tan^{-1}(z)\,dz^n = \frac{z^{n+1}}{\Gamma\left(n+2\right)}\,\mathrm{F}\left(\frac{1}{2},1,1;\frac{n+2}{2},\frac{n+3}{2};-z^2\right)$$

$$I_0 = \tan^{-1}(z)$$

$$I_1 = z\tan^{-1}(z) - \frac{1}{2}\log\left(z^2+1\right)$$

$$I_2 = \frac{z}{2} + \frac{z^2-1}{2}\tan^{-1}(z) - \frac{z}{2}\log\left(z^2+1\right)$$

$$I_{n+1} = \frac{z\,\left(3\,n-1\right)}{n\,\left(n+1\right)}\,I_n - \frac{3\,n\,z^2-2\,z^2+n}{\left(n+1\right)\,n^2}\,I_{n-1} + \frac{z\,\left(z^2+1\right)}{\left(n+1\right)\,n^2}\,I_{n-2}$$

Example.

$$I_n = \int \mathrm{erf}(z)\,dz^n = \frac{2\,z^{n+1}}{\sqrt{\pi}\,\Gamma\left(n+2\right)}\,\mathrm{F}\left(\frac{1}{2},1;\frac{n+2}{2},\frac{n+3}{2};-z^2\right)$$

$$I_0 = \mathrm{erf}(z)$$

$$I_1 = z\,\mathrm{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}$$

$$I_2 = \frac{2\,z^2+1}{4}\,\mathrm{erf}(z) + \frac{z\,e^{-z^2}}{2\,\sqrt{\pi}}$$

$$I_{n+1} = \frac{2\,z}{n+1}\,I_n - \frac{2\,z^2-n}{2\,n\,(n+1)}\,I_{n-1} - \frac{z}{2\,n\,(n+1)}\,I_{n-2}$$

3. An n-Dimensional G Integral

Theorem. *Assume ρ is a positive integer. Then*

$$\int \mathrm{G}\left(\vec{a};\vec{b};\vec{c};\vec{d};\rho\log(z)+\theta\right)\,dz^n$$
$$= \rho^{-n}\,e^{-n\,\theta/\rho}\,\mathrm{G}\left(\Delta(n,\rho),\vec{a}+\frac{n}{\rho};\vec{b}+\frac{n}{\rho};\vec{c}+\frac{n}{\rho};\Delta(0,\rho),\vec{d}+\frac{n}{\rho};\rho\log(z)+\theta\right)$$

4. Definite Integrals

Definition. Let

$$C_n = \{\vec{x}\,|\,0 \le x_i \le 1\}$$
$$C_n^{+}\,\mathrm{F}\,(\vec{x}) = \alpha_1^{-1}\,\alpha_2^{-1}\,\cdots\,\alpha_n^{-1}\,\mathrm{F}\,(\vec{x})\Big|_{x_1=0}^{1}\Big|_{x_2=0}^{1}\cdots\Big|_{x_n=0}^{1}$$

Theorem. *Assume* $\mathrm{F}^{(n)}(z) = \mathrm{f}(z)$*. Then*

$$\int_{C_n} \mathrm{f}\,(\vec{\alpha}\cdot\vec{x}+\beta)\,d\vec{x} = C_n^{+}\,\mathrm{F}\,(\vec{\alpha}\cdot\vec{x}+\beta)$$