Solutions to Selected Dwight Integrals

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This note contains hand solutions to selected integrals from *Tables of Integrals and Other Mathematical Data* by Herbert B. Dwight.

(850.1) $\displaystyle\int_0^\infty z^{n-1}\, e^{-z}\, dz$

$$\int_0^\infty z^{n-1}\, e^{-z}\, dz = \int_0^\infty z^{n-1}\, \mathrm{G}\left(;\,;0;\,;z\right)\, dz = \Gamma(n)$$

(853.21) $\displaystyle\int_0^1 z^{m-1}\, (1-z)^{n-1}\, dz$

$$(1-z)^{n-1}\, \mathrm{H}\left(1-|\,z\,|\right) = \Gamma(n)\, \mathrm{G}\left(;n;0;\,;z\right)$$

$$\int_0^1 z^{m-1}\, (1-z)^{n-1}\, dz = \int_0^\infty z^{m-1}\, \Gamma(n)\, \mathrm{G}\left(;n;0;\,;z\right)\, dz$$

$$= \Gamma(n) \int_0^\infty \mathrm{G}\left(;m+n-1;m-1;\,;z\right)\, dz = \Gamma(n)\, \frac{\Gamma(m)}{\Gamma\left(m+n\right)} = \mathrm{B}(m,n)$$

(855.13) $\displaystyle\int_0^1 \frac{z^p + z^{-p}}{z\, \left(z^q + z^{-q}\right)}\, dz$

$$\int_0^1 \frac{z^p + z^{-p}}{z\, \left(z^q + z^{-q}\right)}\, dz = \int_0^1 \frac{z^p + z^{-p}}{z} \sum_{n=0}^\infty (-1)^n\, z^{(2\,n+1)\,q}\, dz$$

$$= \sum_{n=0}^\infty (-1)^n \int_0^1 \frac{z^{p+(2\,n+1)\,q} + z^{-p+(2\,n+1)\,q}}{z}\, dz$$

$$= \sum_{n=0}^\infty (-1)^n \left(\frac{1}{p+(2\,n+1)\,q} + \frac{1}{-p+(2\,n+1)\,q}\right)$$

$$= \frac{2}{q} \sum_{n=0}^\infty (-1)^n\, \frac{2\,n+1}{(2\,n+1)^2 - \left(\frac{p}{q}\right)^2} = \frac{2}{q} \cdot \frac{\pi}{4} \sec\left(\frac{\pi\, p}{2\,q}\right) = \frac{\pi}{2\,q} \sec\left(\frac{\pi\, p}{2\,q}\right)$$

(855.51) $\displaystyle\int_0^a z^{m-1}\, (a-z)^{n-1}\, dz$

$$\int_0^a z^{m-1}\, (a-z)^{n-1}\, dz = a^{m+n-1} \int_0^1 t^{m-1}\, (1-t)^{n-1}\, dt$$

$$= a^{m+n-1}\, \mathrm{B}(m,n)$$

(856.12) $\displaystyle\int_0^\infty \frac{z^{m-1}}{(a+b\,z)^{m+n}}\,dz$

$$\frac{1}{(a+b\,z)^{m+n}} = a^{-m-n}\left(1+\frac{b\,z}{a}\right)^{-m-n}$$
$$= \frac{a^{-m-n}}{\Gamma\,(m+n)}\,\mathrm{G}\left(-m-n+1;;0;;\frac{b\,z}{a}\right)$$

$$\int_0^\infty \frac{z^{m-1}}{(a+b\,z)^{m+n}}\,dz = \frac{a^{-m-n}}{\Gamma\,(m+n)}\int_0^\infty z^{m-1}\,\mathrm{G}\left(-m-n+1;;0;;\frac{b\,z}{a}\right)\,dz$$
$$= \frac{a^{-m-n}}{\Gamma\,(m+n)}\left(\frac{b}{a}\right)^{-m}\Gamma(n)\,\Gamma(m) = a^{-n}\,b^{-m}\,\frac{\Gamma(m)\,\Gamma(n)}{\Gamma\,(m+n)}$$
$$= a^{-n}\,b^{-m}\,\mathrm{B}(m,n)$$

(858.3(1)) $\displaystyle\int_0^\pi \sin(z)^2\,dz$

$$\int_0^\pi \sin(z)^2\,dz = \int_0^\pi \left(\frac{1}{2}-\frac{1}{2}\cos\,(2\,z)\right)\,dz = \left(\frac{z}{2}-\frac{1}{4}\sin\,(2\,z)\right)\Bigg|_{z=0}^{\pi} = \frac{\pi}{2}$$

(858.45(1)) $\displaystyle\int_0^{\pi/2} \sin(z)^p\,dz$

$$\int_0^{\pi/2} \sin(z)^p\,dz = \frac{1}{2}\,\mathrm{B}\left(\frac{p+1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{p}{2}+1\right)} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}$$
$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\,\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p}{2}+1\right)^2}$$
$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{2\,\pi}\,2^{-2\,(p/2+1/2)+1/2}\,\Gamma\,(p+1)}{\Gamma\left(\frac{p}{2}+1\right)^2} = \frac{\pi\,\Gamma\,(p+1)}{2^{p+1}\,\Gamma\left(\frac{p}{2}+1\right)^2}$$

(858.502) $\displaystyle\int_0^{\pi/2} \cos(z)^p\cos\,(m\,z)\;dz$

$$\frac{\cos\left(m\cos^{-1}(t)\right)}{\sqrt{1-t^2}}\,\mathrm{H}\,(1-|\,t\,|) = \sqrt{\pi}\,\mathrm{G}\left(;\frac{1+m}{2},\frac{1-m}{2};0,\frac{1}{2};;t^2\right)$$

$$\int_0^{\pi/2} \cos(z)^p \cos(m\,z)\;dz = \int_0^1 \frac{t^p \cos\left(m\cos^{-1}(t)\right)}{\sqrt{1-t^2}}\,dt$$
$$= \sqrt{\pi}\int_0^\infty t^p\,\mathrm{G}\left(;\frac{1+m}{2},\frac{1-m}{2};0,\frac{1}{2};;t^2\right)\,dt$$
$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\,\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{1+m}{2}+\frac{p+1}{2}\right)\,\Gamma\left(\frac{1-m}{2}+\frac{p+1}{2}\right)}$$
$$= \frac{\sqrt{\pi}}{2}\cdot\sqrt{2\,\pi}\,2^{-2\,(p/2+1/2)+1/2}\,\frac{\Gamma\left(p+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)\,\Gamma\left(\frac{p-m}{2}+1\right)}$$
$$= \frac{\pi}{2^{p+1}}\cdot\frac{\Gamma\left(p+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)\,\Gamma\left(\frac{p-m}{2}+1\right)}$$

(858.516) $\quad\int_0^\pi \sin\left(m\,z\right)\,\sin\left(n\,z\right)\;dz$

$$\int_0^{\pi/2} \sin(x)^p\cos(x)^q\,dx = \frac{1}{2}\int_0^1 t^{\frac{p-1}{2}}\;(1-t)^{\frac{q-1}{2}}\;dt = \frac{1}{2}\,\mathrm{B}\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$
$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\,\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q}{2}+1\right)}$$

(858.516) $\quad\int_0^\pi \sin\left(m\,z\right)\,\sin\left(n\,z\right)\;dz$

$$\int_0^\pi \sin\left(m\,z\right)\,\sin\left(n\,z\right)\;dz = \int_0^\pi\left(-\frac{1}{2}\cos\left(\left(m+n\right)\;z\right)+\frac{1}{2}\cos\left(\left(m-n\right)\;z\right)\right)\;dz$$
$$= \left(-\frac{1}{2\;(m+n)}\sin\left(\left(m+n\right)\;z\right)+\frac{1}{2\;(m-n)}\sin\left(\left(m-n\right)\;z\right)\right)\Bigg|_{z=0}^{\pi}$$
$$= -\frac{1}{2\;(m+n)}\sin\left(\left(m+n\right)\;\pi\right)+\frac{1}{2\;(m-n)}\sin\left(\left(m-n\right)\;\pi\right)$$

$m\neq n$

$$\int_0^\pi \sin\left(m\,z\right)^2\;dz = \int_0^\pi\left(-\frac{1}{2}\cos\left(2\,m\,z\right)+\frac{1}{2}\right)\;dz = -\frac{1}{2\,m}\sin\left(2\,m\,\pi\right)+\frac{\pi}{2}$$

$m=n$

(858.524(2)) $$\int_0^{\pi} \frac{1}{1 - a\cos(z)}\, dz$$

$$\int_0^{\pi} \frac{1}{1 - a\cos(z)}\, dz = \frac{1}{2} \int_0^{2\,\pi} \frac{1}{1 - a\cos(z)}\, dz = -\frac{i}{2} \oint_C \frac{1}{t\,\left(1 - a\,\frac{t+t^{-1}}{2}\right)}\, dt$$

$$= \frac{i}{a} \oint_C \frac{1}{\left(t + \frac{-1+\sqrt{1-a^2}}{a}\right)\left(t + \frac{-1-\sqrt{1-a^2}}{a}\right)}\, dt = -\frac{2\,\pi}{a} \underset{t=-\frac{-1+\sqrt{1-a^2}}{a}}{\text{residue}} \mathrm{f}(t)$$

$$= -\frac{2\,\pi}{a} \frac{1}{-\frac{-1+\sqrt{1-a^2}}{a} + \frac{-1-\sqrt{1-a^2}}{a}} = \frac{\pi}{\sqrt{1-a^2}}$$

(858.533) $$\int_0^{\pi} \frac{1}{(1 - a\sin(z))^2}\, dz$$

$$\int_0^{\pi} \frac{1}{(1 - a\sin(z))^2}\, dz = -\frac{a\cos(z)\,(1 - a\sin(z))}{1 - a^2}\Bigg|_{z=0}^{\pi} + \frac{1}{1-a^2} \int_0^{\pi} \frac{1}{1 - a\sin(z)}\, dz$$

$$= -\frac{a\cos(z)\,(1 - a\sin(z))}{1 - a^2}\Bigg|_{z=0}^{\pi} + \frac{2}{(1-a^2)^{3/2}} \tan^{-1}\left(\frac{\tan\left(\frac{z}{2}\right) - a}{\sqrt{1-a^2}}\right)\Bigg|_{z=0}^{\pi}$$

$$= \frac{2\,a}{1-a^2} + \frac{\pi}{(1-a^2)^{3/2}} + \frac{2}{(1-a^2)^{3/2}} \tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right)$$

$$= \frac{2\,a}{1-a^2} + \frac{\pi}{(1-a^2)^{3/2}} + \frac{2}{(1-a^2)^{3/2}} \sin^{-1}(a) = \frac{2\,a}{1-a^2} + \frac{\pi + 2\sin^{-1}(a)}{(1-a^2)^{3/2}}$$

(858.540(2)) $$\int_0^{\pi/2} \frac{1}{1 + a^2\cos(z)^2}\, dz$$

$$\int_0^{\pi/2} \frac{1}{1 + a^2\cos(z)^2}\, dz = \int_0^{\pi/2} \frac{1}{1 + \frac{a^2}{2} + \frac{a^2}{2}\cos(2\,z)}\, dz$$

$$= \frac{2}{a^2+2} \int_0^{\pi/2} \frac{1}{1 + \frac{a^2}{a^2+2}\cos(2\,z)}\, dz = \frac{1}{a^2+2} \int_0^{\pi} \frac{1}{1 + \frac{a^2}{a^2+2}\cos(2\,z)}\, dz$$

$$= \frac{1}{a^2+2} \cdot \frac{\pi}{\sqrt{1 - \left(\frac{a^2}{a^2+2}\right)^2}} = \frac{1}{a^2+2} \cdot \frac{\pi}{\sqrt{4\,\frac{a^2+1}{(a^2+2)^2}}} = \frac{\pi}{2\,\sqrt{a^2+1}}$$

$$(858.550) \qquad \int_0^{\pi/2} \frac{1}{a^2 \sin(z)^2 + b^2 \cos(z)^2}\, dz$$

$$\int_0^{\pi/2} \frac{1}{a^2 \sin(z)^2 + b^2 \cos(z)^2}\, dz = \int_0^{\pi/2} \frac{1}{\frac{a^2+b^2}{2} + \frac{-a^2+b^2}{2} \cos(2\, z)}\, dz$$
$$= \frac{2}{a^2+b^2} \int_0^{\pi/2} \frac{1}{1 + \frac{-a^2+b^2}{a^2+b^2} \cos(2\, z)}\, dz = \frac{1}{a^2+b^2} \int_0^{\pi} \frac{1}{1 + \frac{-a^2+b^2}{a^2+b^2} \cos(2\, z)}\, dz$$
$$= \frac{1}{a^2+b^2} \frac{\pi}{\sqrt{1 - \left(\frac{-a^2+b^2}{a^2+b^2}\right)^2}} = \frac{1}{a^2+b^2} \frac{\pi}{\sqrt{4\, \frac{a^2\, b^2}{(a^2+b^2)^2}}} = \frac{\pi}{2\, a\, b}$$

$$(858.560(2)) \qquad \int_0^{\infty} \cos\left(a^2\, z^2\right)\, dz$$

$$\int_0^{\infty} \cos\left(a^2\, z^2\right)\, dz = \int_0^{\infty} \sqrt{\pi}\, \mathrm{G}\left(;;0;\frac{1}{2};\frac{a^4\, z^4}{4}\right)\, dz$$
$$= \sqrt{\pi} \cdot \frac{1}{4} \left(\frac{a^4}{4}\right)^{-1/4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{2}+1-\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\sqrt{2}}{a} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{2^{3/2}\, a}$$

$$(858.630(1)) \qquad \int_0^{\infty} \frac{\sin(m\, z)^2}{z}\, dz$$

$$\int_0^{\infty} \frac{\sin(m\, z)^2}{z}\, dz = \left(\frac{1}{2} \log(m\, z) - \frac{1}{2} \mathrm{Ci}(2\, m\, z)\right)\Bigg|_{z=0}^{\infty}$$
$$= \left(\frac{1}{2} \log(m\, z) - \frac{\gamma}{2} - \frac{1}{2} \log(2\, m\, z) + m^2\, z^2\, {}_2\mathrm{F}_3\left(1,1;\frac{3}{2},2,2;m^2\, z^2\right)\right)\Bigg|_{z=0}^{\infty}$$
$$= \left(-\frac{\gamma}{2} - \frac{1}{2} \log(2) + m^2\, z^2\, {}_2\mathrm{F}_3\left(1,1;\frac{3}{2},2,2;m^2\, z^2\right)\right)\Bigg|_{z=0}^{\infty} = \infty$$

$$(858.653) \qquad \int_0^{\infty} \frac{\sin(m\, z)^3}{z^2}\, dz$$

$$\int_0^{\infty} \frac{\sin(m\, z)^3}{z^2}\, dz$$
$$= \left(\frac{\sin(3\, m\, z)}{4\, z} - \frac{3\, m}{4} \mathrm{Ci}(3\, m\, z) - \frac{3}{4\, z} \sin(m\, z) + \frac{3\, m}{4} \mathrm{Ci}(m\, z)\right)\Bigg|_{z=0}^{\infty}$$
$$= \lim_{z\to\infty} \left(\frac{\sin(3\, m\, z)}{4\, z} - \mathrm{O}\left(\frac{1}{z}\right) - \frac{3 \sin(m\, z)}{4\, z} + \mathrm{O}\left(\frac{1}{z}\right)\right)$$
$$- \lim \left(\frac{\sin(3\, m\, z)}{4\, z} - \frac{3 \sin(m\, z)}{4\, z}\right)$$
$$= 0 - \lim \left(\frac{3\, m}{4} + \frac{3\, m}{4} \mathrm{Log}\left(\frac{1}{3}\right) - \frac{3\, m}{4}\right) = \frac{3\, m}{4} \log(3)$$

(858.713) $\displaystyle\int_0^\infty \frac{\sin(a\,z)^2\,\cos(m\,z)}{z^2}\,dz$

$m > 2\,a > 0$

$$\begin{aligned}
&\int_0^\infty \frac{\sin(a\,z)^2\,\cos(m\,z)}{z^2}\,dz \\
&\quad= \left(\frac{\cos((m-2\,a)\;z)}{4\,z} - \frac{\cos(m\,z)}{2\,z} + \frac{\cos((m+2\,a)\;z)}{4\,z}\right. \\
&\qquad\left. + \frac{m-2\,a}{4}\,\mathrm{Si}((m-2\,a)\;z) - \frac{m}{2}\,\mathrm{Si}(m\,z) + \frac{m+2\,a}{4}\,\mathrm{Si}((m+2\,a)\;z)\right)\Bigg|_{z=0}^{\infty} \\
&\quad= \lim_{z\to\infty}\left(\frac{m-2\,a}{4}\cdot\frac{\pi}{2} - \frac{m}{2}\cdot\frac{\pi}{2} + \frac{m+2\,a}{4}\cdot\frac{\pi}{2}\right) = -\lim_{z\to 0}\left(\frac{1}{4\,z} - \frac{1}{2\,z} + \frac{1}{4\,z}\right) = 0
\end{aligned}$$

$2\,a > m > 0$

$$\begin{aligned}
&\int_0^\infty \frac{\sin(a\,z)^2\,\cos(m\,z)}{z^2}\,dz = \lim_{z\to\infty}\left(\frac{m-2\,a}{4}\cdot\left(-\frac{\pi}{2}\right) - \frac{m}{2}\cdot\frac{\pi}{2} + \frac{m+2\,a}{4}\cdot\frac{\pi}{2}\right) \\
&\quad= \frac{2\,a-m}{4}\,\pi
\end{aligned}$$

(858.821) $\displaystyle\int_0^\infty \frac{\sin(m\,z)\;\cos(n\,z)}{\sqrt{z}}\,dz$

$n > m > 0$

$$\begin{aligned}
&\int_0^\infty \frac{\sin(m\,z)\;\cos(n\,z)}{\sqrt{z}}\,dz \\
&\quad= \left(\sqrt{\frac{\pi}{2\;(m+n)}}\,\mathrm{S}\left(\sqrt{\frac{2\,m+2\,n}{\pi}}\,\sqrt{z}\right) - \sqrt{\frac{\pi}{2\;(n-m)}}\,\mathrm{S}\left(\sqrt{\frac{2\,n-2\,m}{\pi}}\,\sqrt{z}\right)\right)\Bigg|_{z=0}^{\infty} \\
&\quad= \lim_{z\to\infty}\left(\sqrt{\frac{\pi}{2\;(m+n)}}\left(\frac{1}{2} + \mathrm{O}\left(\frac{1}{\sqrt{z}}\right)\right) - \sqrt{\frac{\pi}{2\;(n-m)}}\left(\frac{1}{2} + \mathrm{O}\left(\frac{1}{\sqrt{z}}\right)\right)\right) \\
&\quad= \sqrt{\frac{\pi}{8}}\left(\frac{1}{\sqrt{m+n}} - \frac{1}{\sqrt{n-m}}\right)
\end{aligned}$$

(859.002) $\displaystyle\int_0^\infty \frac{\sin(m\,z)^2}{a^2+z^2}\,dz$

$$\begin{aligned}
&\int_0^\infty \frac{\sin(m\,z)^2}{a^2+z^2}\,dz = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin(m\,z)^2}{a^2+z^2}\,dz \\
&\quad= \frac{\pi\,i}{4}\left(-\underset{z=i\,a}{\mathrm{residue}}\,\frac{e^{2\,i\,m\,z}}{a^2+z^2} + \underset{z=i\,a}{\mathrm{residue}}\,\frac{2}{a^2+z^2} + \underset{z=-i\,a}{\mathrm{residue}}\,\frac{e^{-2\,i\,m\,z}}{a^2+z^2}\right) \\
&\quad= \frac{\pi\,i}{4}\left(-2\,\frac{e^{-2\,a\,m}}{2\,i\,a} + \frac{2}{2\,i\,a}\right) = \frac{\pi}{4\,a}\,\left(1-e^{-2\,a\,m}\right)
\end{aligned}$$

(859.014) $\displaystyle \int_0^\infty \frac{\sin(m\,z)}{z\,\left(a^2+z^2\right)^2}\,dz$

$$\int_0^\infty \frac{\sin(m\,z)}{z\,\left(a^2+z^2\right)^2}\,dz = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin(m\,z)}{z\,\left(a^2+z^2\right)^2}\,dz$$
$$= \frac{\pi}{2}\left(\operatorname*{residue}_{z=i\,a}\frac{e^{i\,m\,z}-1}{z\,\left(a^2+z^2\right)^2} + \operatorname*{residue}_{z=-i\,a}\frac{e^{-i\,m\,z}-1}{z\,\left(a^2+z^2\right)^2}\right)$$
$$= \frac{\pi}{2}\left(\frac{2\,e^{a\,m}-2-a\,m}{4\,a^4\,e^{a\,m}} + \frac{2\,e^{a\,m}-2-a\,m}{4\,a^4\,e^{a\,m}}\right) = \frac{\pi}{2\,a^4}\left(1-\frac{2+a\,m}{2}\,e^{-a\,m}\right)$$

(859.124) $\displaystyle \int_0^\pi \frac{a-b\cos(z)}{a^2-2\,a\,b\cos(z)+b^2}\,dz$

$a > b > 0$

$$\int_0^\pi \frac{a-b\cos(z)}{a^2-2\,a\,b\cos(z)+b^2}\,dz = \frac{1}{2}\int_0^{2\,\pi} \frac{a-b\cos(z)}{a^2-2\,a\,b\cos(z)+b^2}\,dz$$
$$= -\frac{i}{4}\oint_C \frac{2\,a-b\,\left(t+t^{-1}\right)}{t\,\left(a^2-a\,b\,\left(t+t^{-1}\right)+b^2\right)}\,dt$$
$$= -\frac{i}{4}\oint_C \left(\frac{1}{a\,t}+\frac{1}{a\,t-b}-\frac{b}{a\,(b\,t-a)}\right)dt$$
$$= -\frac{i}{4}\cdot 2\,\pi\,i\left(\operatorname*{residue}_{t=0}\,\mathrm{f}(t) + \operatorname*{residue}_{t=b/a}\,\mathrm{f}(t)\right) = \frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{a}\right) = \frac{\pi}{a}$$

$b > a > 0$

$$\int_0^\pi \frac{a-b\cos(z)}{a^2-2\,a\,b\cos(z)+b^2}\,dz = -\frac{i}{4}\cdot 2\,\pi\,i\left(\operatorname*{residue}_{t=0}\,\mathrm{f}(t) + \operatorname*{residue}_{t=a/b}\,\mathrm{f}(t)\right) = \frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{a}\right)$$
$$= 0$$

(859.165) $$\int_0^\infty \frac{1}{1-2\,z^2\cos(\phi)+z^4}\,dz$$

$0<\phi<\pi$

$$\int_0^\infty \frac{1}{1-2\,z^2\cos(\phi)+z^4}\,dz = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{1-2\,z^2\cos(\phi)+z^4}\,dz$$
$$= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{\left(z+e^{i\,\phi/2}\right)\left(z-e^{i\,\phi/2}\right)\left(z+e^{-i\,\phi/2}\right)\left(z-e^{-i\,\phi/2}\right)}\,dz$$
$$= \pi\,i\left(\underset{z=e^{i\,\phi/2}}{\text{residue}}\ \mathrm{f}(z) + \underset{z=-e^{-i\,\phi/2}}{\text{residue}}\ \mathrm{f}(z)\right)$$
$$= \pi\,i$$
$$\times\left(\frac{1}{2\,e^{i\,\phi/2}\left(e^{i\,\phi/2}+e^{-i\,\phi/2}\right)\left(e^{i\,\phi/2}-e^{-i\,\phi/2}\right)}\right.$$
$$\left.-\frac{1}{2\left(e^{i\,\phi/2}-e^{-i\,\phi/2}\right)\left(-e^{-i\,\phi/2}-e^{i\,\phi/2}\right)e^{-i\,\phi/2}}\right)$$
$$= \frac{\pi}{4\sin\left(\frac{\phi}{2}\right)}$$

(860.12) $$\int_0^\infty z\,e^{-r^2\,z^2}\,dz$$

$$\int_0^\infty z\,e^{-r^2\,z^2}\,dz = \int_0^\infty z\,\mathrm{G}\left(;;0;;r^2\,z^2\right)\,dz = \frac{1}{2}\left(r^2\right)^{-1}\Gamma(1) = \frac{1}{2\,r^2}$$

(860.24) $$\int_0^\infty \frac{1-e^{-a\,z^2}}{z^2}\,dz$$

$$\int_0^\infty \frac{1-e^{-a\,z^2}}{z^2}\,dz = \left(-\frac{1}{z}+\frac{e^{-a\,z^2}}{z}+\sqrt{\pi\,a}\,\mathrm{erf}\left(\sqrt{a}\,z\right)\right)\Bigg|_{z=0}^{\infty}$$
$$= \lim_{z\to\infty}\left(-\frac{1}{z}+\frac{e^{-a\,z^2}}{z}+\sqrt{\pi\,a}\left(1+\mathrm{O}\left(\frac{e^{-a\,z^2}}{z}\right)\right)\right)$$
$$-\lim_{z\to 0}\left(-\frac{1}{z}+\frac{1}{z}-a\,z+\mathrm{O}\left(z^3\right)+\sqrt{\pi\,a}\left(\frac{2\sqrt{a}}{\sqrt{\pi}}\,z\right)\right)$$
$$= \sqrt{\pi\,a}$$

(860.39) $\displaystyle\int_0^\infty \frac{z^{p-1}}{e^{a\,z}-1}\,dz$

$$\begin{aligned}
\int_0^\infty \frac{z^{p-1}}{e^{a\,z}-1}\,dz &= \int_0^\infty z^{p-1}\sum_{n=1}^\infty e^{-n\,a\,z}\,dz = \sum_{n=1}^\infty \int_0^\infty z^{p-1}\,e^{-n\,a\,z}\,dz \\
&= \sum_{n=1}^\infty \int_0^\infty z^{p-1}\,\mathrm{G}\left(;;0;;n\,a\,z\right)\,dz = \sum_{n=1}^\infty \frac{1}{(n\,a)^p}\,\Gamma(p) = \frac{\Gamma(p)}{a^p}\sum_{n=1}^\infty \frac{1}{n^p} \\
&= \frac{\Gamma(p)}{a^p}\,\zeta(p)
\end{aligned}$$

(860.41) $\displaystyle\int_0^\infty \frac{z}{e^{a\,z}+1}\,dz$

$$\begin{aligned}
\int_0^\infty \frac{z}{e^{a\,z}+1}\,dz &= \int_0^\infty z\sum_{n=0}^\infty (-1)^n\;e^{-(n+1)\,a\,z}\,dz \\
&= \sum_{n=0}^\infty (-1)^n \int_0^\infty z\,e^{-(n+1)\,a\,z}\,dz = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2\;a^2}\int_0^\infty t\,e^{-t}\,dt \\
&= \frac{1}{a^2}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2} = \frac{1}{a^2}\left(1-\frac{2}{2^2}\right)\zeta(2) = \frac{1}{a^2}\cdot\frac{1}{2}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{12\,a^2}
\end{aligned}$$

(860.504) $\displaystyle\int_0^\infty \frac{z^4}{\sinh(a\,z)}\,dz$

$$\int_0^\infty \frac{z^4}{\sinh(a\,z)}\,dz = \frac{2\,\Gamma(5)}{a^5}\left(1-\frac{1}{2^5}\right)\zeta(5) = \frac{93}{2\,a^5}\,\zeta(5)$$

(860.509) $\displaystyle\int_0^\infty \frac{z^{p-1}}{\sinh(a\,z)}\,dz$

$$\begin{aligned}
\int_0^\infty \frac{z^{p-1}}{\sinh(a\,z)}\,dz &= 2\int_0^\infty \frac{z^{p-1}}{e^{a\,z}-e^{-a\,z}}\,dz = 2\int_0^\infty \frac{z^{p-1}\,e^{-a\,z}}{1-e^{-2\,a\,z}}\,dz \\
&= 2\int_0^\infty z^{p-1}\,e^{-a\,z}\sum_{n=0}^\infty e^{-2\,n\,a\,z}\,dz = 2\sum_{n=0}^\infty \int_0^\infty z^{p-1}\,e^{-(2\,n+1)\,a\,z}\,dz \\
&= 2\sum_{n=0}^\infty \frac{1}{(2\,n+1)^p\;a^p}\int_0^\infty t^{p-1}\,e^{-t}\,dt = \frac{2\,\Gamma(p)}{a^p}\sum_{n=0}^\infty \frac{1}{(2\,n+1)^p} \\
&= \frac{2\,\Gamma(p)}{a^p}\left(1-\frac{1}{2^p}\right)\zeta(p)
\end{aligned}$$

(860.518) $$\int_0^\infty \frac{z^{2\,n}}{\sinh{(a\,z)}^2}\,dz$$

$$\int_0^\infty \frac{z^{2\,n}}{\sinh{(a\,z)}^2}\,dz = \frac{\Gamma\,(2\,n+1)}{2^{2\,n-1}\,a^{2\,n+1}}\,\zeta\,(2\,n)$$
$$= \frac{\Gamma\,(2\,n+1)}{2^{2\,n-1}\,a^{2\,n+1}}\cdot\frac{(2\,\pi)^{2\,n}}{2\;(2\,n)!}\,B_{2\,n} = \frac{\pi^{2\,n}}{a^{2\,n+1}}\,B_{2\,n}$$

(860.519) $$\int_0^\infty \frac{z^{p-1}}{\sinh{(a\,z)}^2}\,dz$$

$$\int_0^\infty \frac{z^{p-1}}{\sinh{(a\,z)}^2}\,dz = 4\int_0^\infty \frac{z^p}{(e^{a\,z}-e^{-a\,z})^2}\,dz = 4\int_0^\infty \frac{z^p\,e^{-2\,a\,z}}{(1-e^{-2\,a\,z})^2}\,dz$$
$$= 4\int_0^\infty z^p\,e^{-2\,a\,z}\sum_{n=0}^\infty \binom{-2}{n}\left(-e^{-2\,a\,z}\right)^n\,dz$$
$$= 4\int_0^\infty z^p\,e^{-2\,a\,z}\sum_{n=0}^\infty (n+1)\;e^{-2\,n\,a\,z}\,dz$$
$$= 4\sum_{n=0}^\infty (n+1)\int_0^\infty z^p\,e^{-2\,(n+1)\,a\,z}\,dz$$
$$= 4\sum_{n=0}^\infty (n+1)\;\frac{1}{(2\,a\;(n+1))^{p+1}}\int_0^\infty t^p\,e^{-t}\,dt$$
$$= \frac{\Gamma\,(p+1)}{2^{p-1}\,a^{p+1}}\sum_{n=0}^\infty \frac{1}{(n+1)^p} = \frac{\Gamma\,(p+1)}{2^{p-1}\,a^{p+1}}\,\zeta(p)$$

(860.549) $$\int_0^\infty \frac{z^p}{\cosh{(a\,z)}^2}\,dz$$

$$\int_0^\infty \frac{z^p}{\cosh{(a\,z)}^2}\,dz = 4\int_0^\infty \frac{z^p}{(e^{a\,z}+e^{-a\,z})^2}\,dz$$
$$= 4\int_0^\infty z^p\,e^{-2\,a\,z}\sum_{n=0}^\infty \binom{-2}{n}\left(e^{-2\,a\,z}\right)^n\,dz$$
$$= 4\int_0^\infty z^p\,e^{-2\,a\,z}\sum_{n=0}^\infty (-1)^n\;(n+1)\;e^{-2\,n\,a\,z}\,dz$$
$$= 4\sum_{n=0}^\infty (-1)^n\;(n+1)\int_0^\infty z^p\,e^{-2\,(n+1)\,a\,z}\,dz$$
$$= 4\sum_{n=0}^\infty (-1)^n\;(n+1)\;\frac{1}{(2\,a\;(n+1))^{p+1}}\int_0^\infty t^p\,e^{-t}\,dt$$
$$= \frac{\Gamma\,(p+1)}{2^{p-1}\,a^{p+1}}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^p} = \frac{\Gamma\,(p+1)}{2^{p-1}\,a^{p+1}}\left(1-\frac{1}{2^{p-1}}\right)\zeta(p)$$

(860.82) $\displaystyle\int_0^\infty z^2\, e^{-a\,z} \sin(m\,z)\ dz$

$$\int_0^\infty z^2\, e^{-a\,z} \sin(m\,z)\ dz$$
$$= \frac{1}{2\,i} \left(\int_0^\infty z^2\, e^{-a\,z+i\,m\,z}\, dz - \int_0^\infty z^2\, e^{-a\,z-i\,m\,z}\, dz \right)$$
$$= \frac{1}{2\,i} \left(\frac{2}{(a-i\,m)^3} - \frac{2}{(a+i\,m)^3} \right) = \frac{2\,m\ \left(3\,a^2-m^2\right)}{\left(m^2+a^2\right)^3}$$

(860.90) $\displaystyle\int_0^\infty e^{-a\,z} \cos(m\,z)\ dz$

$$\int_0^\infty e^{-a\,z} \cos(m\,z)\ dz = \frac{1}{m}\, \mathrm{G}\left(\frac{1}{2};;\frac{1}{2};;\frac{a^2}{m^2}\right) = \frac{a}{m^2}\, \mathrm{G}\left(0;;0;;\frac{a^2}{m^2}\right)$$
$$= \frac{a}{m^2} \left(1+\frac{a^2}{m^2}\right)^{-1} = \frac{a}{m^2+a^2}$$

(861.05) $\displaystyle\int_0^\infty \frac{\left(e^{-a\,z}-e^{-b\,z}\right)\ \cos(m\,z)}{z}\, dz$

$$\frac{e^{-a\,z}-1}{z} = \sum_{n=0}^\infty \frac{(-a\,z)^{n+1}}{z\ (n+1)!} = -a \sum_{n=0}^\infty \frac{(-a\,z)^n}{(n+1)\ n!} = -a \sum_{n=0}^\infty \Gamma\binom{n+1}{n+2} \frac{(-a\,z)^n}{n!}$$
$$= -a\, {}_1\mathrm{F}_1\,(1;2;-a\,z)$$

$$\int_0^\infty \frac{\left(e^{-a\,z}-1\right)\ \cos(m\,z)}{z}\, dz = -a \int_0^\infty {}_1\mathrm{F}_1\,(1;2;-a\,z)\ \cos(m\,z)\ dz$$
$$= -\frac{a}{2\,m}\, \mathrm{G}\left(\frac{1}{2},\frac{1}{2};;\frac{1}{2};-\frac{1}{2};\frac{a^2}{m^2}\right) = -\frac{1}{2}\, \mathrm{G}\left(1,1;;1;0;\frac{a^2}{m^2}\right) = -\frac{1}{2}\log\left(1+\frac{a^2}{m^2}\right)$$

$$\int_0^\infty \frac{\left(e^{-a\,z}-e^{-b\,z}\right)\ \cos(m\,z)}{z}\, dz = -\frac{1}{2}\log\left(1+\frac{a^2}{m^2}\right) + \frac{1}{2}\log\left(1+\frac{b^2}{m^2}\right)$$
$$= \frac{1}{2}\log\left(\frac{b^2+m^2}{m^2+a^2}\right)$$

(861.21) $\displaystyle\int_0^\infty z\, e^{-a^2\,z^2} \sin(m\,z)\ dz$

$$\int_0^\infty z\, e^{-a^2\,z^2} \sin(m\,z)\ dz = \int_0^\infty z\, \mathrm{G}\left(;;0;;a^2\,z^2\right)\ \sqrt{\pi}\, \mathrm{G}\left(;;\frac{1}{2};0;\frac{m^2\,z^2}{4}\right)\ dz$$
$$= \frac{\sqrt{\pi}}{2\,a^2}\, \mathrm{G}\left(;;\frac{1}{2};;\frac{m^2}{4\,a^2}\right) = \frac{\sqrt{\pi}\,m}{4\,a^3}\, \mathrm{G}\left(;;0;;\frac{m^2}{4\,a^2}\right) = \frac{\sqrt{\pi}\,m}{4\,a^3}\, e^{-m^2/4\,a^2}$$

(861.62) $\displaystyle\int_0^\infty \frac{\cos(m\,z)}{\cosh(a\,z)}\,dz$

$$\int_0^\infty \frac{\cos(m\,z)}{\cosh(a\,z)}\,dz = 2\int_0^\infty \sum_{n=0}^\infty (-1)^n\; e^{-(2\,n+1)\,a\,z}\cos(m\,z)\;dz$$
$$= 2\sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-(2\,n+1)\,a\,z}\cos(m\,z)\;dz = 2\sum_{n=0}^\infty \frac{(-1)^n\;(2\,n+1)\;a}{(2\,n+1)^2\;a^2+m^2}$$
$$= \frac{2}{a}\sum_{n=0}^\infty \frac{(-1)^n\;(2\,n+1)}{(2\,n+1)^2+\left(\frac{m}{a}\right)^2} = \frac{\pi}{2\,a}\operatorname{sech}\left(\frac{m\,\pi}{2\,a}\right)$$

(861.83) $\displaystyle\int_0^\infty \frac{\tanh(a\,z)\;\cos(m\,z)}{z}\,dz$

$$\int_0^\infty \frac{\tanh(a\,z)\;\cos(m\,z)}{z}\,dz = \int_0^\infty \frac{e^{a\,z}-e^{-a\,z}}{e^{a\,z}+e^{-a\,z}}\,\frac{\cos(m\,z)}{z}\,dz$$
$$= \int_0^\infty \frac{\cos(m\,z)}{z}\;\left(1-e^{-2\,a\,z}\right)\sum_{n=0}^\infty (-1)^n\;e^{-2\,n\,a\,z}\,dz$$
$$= \sum_{n=0}^\infty (-1)^n \int_0^\infty \frac{\cos(m\,z)}{z}\;\left(e^{-2\,n\,a\,z}-e^{-2\,(n+1)\,a\,z}\right)\;dz$$
$$= \frac{1}{2}\sum_{n=0}^\infty (-1)^n\;\log\left(\frac{4\;(n+1)^2\;a^2+m^2}{4\,n^2\,a^2+m^2}\right)$$
$$= \frac{1}{2}\log\left(\frac{4\,a^2+m^2}{m^2}\right)+\frac{1}{2}\log\left(\frac{4\,a^2+m^2}{16\,a^2+m^2}\right)+\frac{1}{2}\log\left(\frac{36\,a^2+m^2}{16\,a^2+m^2}\right)+\frac{1}{2}\log\left(\frac{36\,a^2+m^2}{64\,a^2+m^2}\right)$$
$$+\ldots$$
$$= \log\left(\frac{2\,a}{m}\right)+\log\left(\prod_{n=0}^\infty\left(1+\left(\frac{m}{2\;(2\,n+1)\;a}\right)^2\right)\right)-\log\left(\prod_{n=1}^\infty\left(1+\left(\frac{m}{2\;(2\,n)\;a}\right)^2\right)\right)$$
$$+\log\left(\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot\ldots}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot\ldots}\right)$$
$$= \log\left(\frac{2\,a}{m}\right)+\log\left(\cosh\left(\frac{m\,\pi}{4\,a}\right)\right)-\log\left(\frac{\sinh\left(\frac{m\,\pi}{4\,a}\right)}{\frac{m\,\pi}{4\,a}}\right)+\log\left(\frac{2}{\pi}\right)=\log\left(\coth\left(\frac{m\,\pi}{4\,a}\right)\right)$$

(863.05) $\displaystyle\int_0^1 \sqrt{\log\left(\frac{1}{z}\right)}\,dz$

$$\int_0^1 \sqrt{\log\left(\frac{1}{z}\right)}\,dz = \int_0^\infty \sqrt{t}\,e^{-t}\,dt = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

(863.32) $$\int_0^1 \frac{z\log\left(\frac{1}{z}\right)}{1-z^2}\,dz$$

$$\int_0^1 \frac{z\log\left(\frac{1}{z}\right)}{1-z^2}\,dz = \int_0^\infty \frac{t\,e^{-2\,t}}{1-e^{-2\,t}}\,dt = \int_0^\infty t\,e^{-2\,t}\sum_{n=0}^\infty e^{-2\,n\,t}\,dt$$
$$= \sum_{n=0}^\infty \int_0^\infty t\,e^{-(2\,n+2)\,t}\,dt = \sum_{n=0}^\infty \frac{1}{(2\,n+2)^2} = \frac{1}{4}\,\zeta(2) = \frac{1}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{24}$$

(863.52) $$\int_0^1 \frac{z^p - z^q}{\log(z)}\,dz$$

$$\int_0^1 \frac{z^p - z^q}{\log(z)}\,dz = -\int_0^\infty \frac{e^{-(p+1)\,t} - e^{-(q+1)\,t}}{t}\,dt$$
$$= \left(\mathrm{E}_1\left((p+1)\ t\right) - \mathrm{E}_1\left((q+1)\ t\right)\right)\Big|_{t=0}^{\infty}$$
$$= \lim_{t\to\infty}\left(\mathrm{O}\left(\frac{e^{-(p+1)\,t}}{t}\right) - \mathrm{O}\left(\frac{e^{-(q+1)\,t}}{t}\right)\right)$$
$$-\lim\left(\left(-\gamma + \log\left(\frac{1}{(p+1)\ t}\right)\right) - \left(-\gamma + \log\left(\frac{1}{(q+1)\ t}\right)\right)\right)$$
$$= \log\left(\frac{p+1}{q+1}\right)$$

(863.82) $$\int_0^1 z\log\left(1+z\right)\,dz$$

$$\log\left(1+z\right) = \mathrm{G}\left(1,1;;1;0;z\right)$$

$$z\,\mathrm{H}\left(1-|\,z\,|\right) = \mathrm{G}\left(;2;1;;z\right)$$

$$\int_0^1 z\log\left(1+z\right)\,dz = \int_0^\infty \mathrm{G}\left(;2;1;;z\right)\,\mathrm{G}\left(1,1;;1;0;z\right)\,dz$$
$$= \mathrm{G}\left(-1;2,0;1,-1,-1;;1\right) = \frac{1}{2\,i\,\pi}\oint \Gamma\begin{pmatrix}2+s,1-s,-1-s,-1-s\\ 2-s,-s\end{pmatrix}\,ds$$
$$= \frac{-1}{2\,i}\oint \frac{1}{(1-s^2)\ \sin\left((1-s)\ \pi\right)}\,ds$$
$$= \pi\left(-\frac{1}{4\,\pi} + \frac{1}{\pi} - \frac{1}{4\,\pi} - \frac{1}{3\,\pi} + \frac{1}{8\,\pi} - \frac{1}{15\,\pi} + \frac{1}{24\,\pi} - \dots\right)$$
$$= \frac{1}{2} - \left(\frac{1}{3} - \frac{1}{8} + \frac{1}{15} - \frac{1}{24} + \dots\right) = \frac{1}{2} - \sum_{n=2}^\infty \frac{(-1)^n}{n^2-1} = \frac{1}{4}$$

(864.03) $\displaystyle \int_0^1 \frac{\log\left(\frac{1+z}{z}\right)}{1+z^2}\,dz$

$$\int_0^1 \frac{\log(z)}{1+z^2}\,dz = -\int_0^\infty \frac{t\,e^{-t}}{1+e^{-2\,t}}\,dt = -\int_0^\infty t\,e^{-t}\sum_{n=0}^\infty (-1)^n\; e^{-2\,n\,t}\,dt$$
$$= -\sum_{n=0}^\infty (-1)^n \int_0^\infty t\,e^{-(2\,n+1)\,t}\,dt = -\sum_{n=0}^\infty \frac{(-1)^n}{(2\,n+1)^2} = -G$$

$$\int_0^\infty \frac{\log(1+z)}{1+z^2}\,dz = G + \frac{\pi}{4}\log(2)$$

$$\int_0^1 \frac{\log\left(\frac{1+z}{z}\right)}{1+z^2}\,dz = \int_1^\infty \frac{\log(1+t)}{1+t^2}\,dt = \int_0^1 \frac{\log(1+t)}{1+t^2}\,dt - \int_0^1 \frac{\log(t)}{1+t^2}\,dt$$
$$= \frac{1}{2}\int_0^\infty \frac{\log(1+t)}{1+t^2}\,dt - \frac{1}{2}\int_0^1 \frac{\log(t)}{1+t^2}\,dt$$
$$= \frac{1}{2}\left(G + \frac{\pi}{4}\log(2)\right) - \frac{1}{2}\,(-G) = G + \frac{\pi}{8}$$

(864.52) $\displaystyle \int_0^\infty \frac{z^{p-1}\log\left(\frac{1}{z}\right)}{1+z}\,dz$

$$\int_0^\infty \frac{z^{p-1}\log\left(\frac{1}{z}\right)}{1+z}\,dz = -\frac{\partial}{\partial\epsilon}\int_0^\infty \frac{z^{p-1+\epsilon}}{1+z}\,dz\bigg|_{\epsilon=0} = -\frac{\partial}{\partial\epsilon}\int_0^\infty z^{p-1+\epsilon}\,\mathrm{G}\,(0;;0;;z)\;dz\bigg|_{\epsilon=0}$$
$$= -\frac{\partial}{\partial\epsilon}\Gamma(1-p-\epsilon)\;\Gamma(p+\epsilon)\bigg|_{\epsilon=0} = -\frac{\partial}{\partial\epsilon}\Gamma(1-p-\epsilon)\;\Gamma(p+\epsilon)\bigg|_{\epsilon=0}$$
$$= -\frac{\partial}{\partial\epsilon}\pi\csc(\pi\;(p+\epsilon))\bigg|_{\epsilon=0} = \frac{\pi^2\cos(\pi\,p)}{\sin(\pi\,p)^2}$$

(864.73) $\displaystyle \int_0^\infty \frac{\log(1+z^p)}{z^q}\,dz$

$\log(1+z^p) = \mathrm{G}\,(1,1;;1;0;z^p)$

$$\int_0^\infty \frac{\log(1+z^p)}{z^q}\,dz = \int_0^\infty z^{-q}\,\mathrm{G}\,(1,1;;1;0;z^p)\;dz = \frac{\Gamma\left(-\frac{1-q}{p}\right)^2\,\Gamma\left(1+\frac{1-q}{p}\right)}{p\,\Gamma\left(1-\frac{1-q}{p}\right)}$$
$$= \frac{1}{q-1}\,\Gamma\left(\frac{q-1}{p}\right)\,\Gamma\left(1+\frac{1-q}{p}\right) = \frac{\pi}{q-1}\csc\left(\frac{q-1}{p}\,\pi\right)$$

(865.21) $\displaystyle\int_0^{\pi/2} \sin(z)\log(\sin(z))\, dz$

$$\int_0^{\pi/2} \sin(z)\log(\sin(z))\, dz = \frac{\partial}{\partial\epsilon}\int_0^{\pi/2} \sin(z)^{1+\epsilon}\, dz\Big|_{\epsilon=0} = \frac{\partial}{\partial\epsilon}\frac{1}{2}\,\mathrm{B}\left(1+\frac{\epsilon}{2},\frac{1}{2}\right)\Big|_{\epsilon=0}$$

$$= \frac{\partial}{\partial\epsilon}\frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{2}+\frac{\epsilon}{2}\right)}\Big|_{\epsilon=0} = \frac{\sqrt{\pi}}{2}\left(\frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\,\psi\left(1+\frac{\epsilon}{2}\right)}{2\,\Gamma\left(\frac{3}{2}+\frac{\epsilon}{2}\right)} - \frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\,\psi\left(\frac{3}{2}+\epsilon\right)}{2\,\Gamma\left(\frac{3}{2}+\frac{\epsilon}{2}\right)}\right)\Big|_{\epsilon=0}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\Gamma\left(\frac{3}{2}\right)}\cdot\left(\psi(1)-\psi\left(\frac{3}{2}\right)\right) = \frac{1}{2}\,(-2+2\log(2)) = -1+\log(2)$$

(865.33) $\displaystyle\int_0^{\pi/2} \log(1+\tan(z))\, dz$

$$\int_0^{\pi/2} \log(1+\tan(z))\, dz = \int_0^{\infty}\frac{\log(1+y)}{1+y^2}\, dy$$

$$= \int_0^{\infty} \mathrm{G}\left(1,1;;1;0;y\right)\,\mathrm{G}\left(0;;0;;y^2\right)\, dy$$

$$= \frac{1}{2\,\pi}\int_0^{\infty} \mathrm{G}\left(\frac{1}{2},1,1;;\frac{1}{2},1;0;y^2\right)\,\mathrm{G}\left(0;;0;;y^2\right)\, dy$$

$$= \frac{1}{4\,\pi}\,\mathrm{G}\left(0,0,-\frac{1}{2};\frac{1}{2};0,0,-\frac{1}{2},-\frac{1}{2};;1\right)$$

$$= \frac{1}{4\,\pi}\cdot\frac{1}{2\,i\,\pi}\oint\Gamma\left(\begin{matrix}\frac{3}{2}+s,1+s,1+s,-\frac{1}{2}-s,-\frac{1}{2}-s,-s,-s\\ \frac{1}{2}-s\end{matrix}\right)ds$$

$$= -\frac{1}{4\,\pi}\sum_{n=0}^{\infty}\operatorname*{residue}_{s=n}\frac{\pi^3}{\left(s+\frac{1}{2}\right)\cos(\pi\,s)\,\sin(\pi\,s)^2}$$

$$-\frac{1}{4\,\pi}\sum_{n=0}^{\infty}\operatorname*{residue}_{s=n+1/2}\frac{\pi^3}{\left(s+\frac{1}{2}\right)\cos(\pi\,s)\,\sin(\pi\,s)^2}$$

$$= -\frac{1}{4\,\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\,4\,\pi}{(2\,n+1)^2} - \frac{1}{4\,\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\,\pi^2}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2\,n+1)^2} + \sum_{n=0}^{\infty}\frac{(-1)^n\,\pi}{n+1} = G+\frac{\pi}{4}\log(2)$$

(865.43(1)) $\displaystyle\int_0^\pi \log\left(1+\cos(z)\right)\,dz$

$$\int_0^\pi \log\left(1+\cos(z)\right)\,dz = \int_0^\pi \log\left(2\cos\left(\frac{z}{2}\right)^2\right)\,dz = \pi\log(2) + 2\int_0^\pi \log\left(\cos\left(\frac{z}{2}\right)\right)\,dz$$
$$= \pi\log(2) + 4\int_0^{\pi/2} \log\left(\cos(t)\right)\,dt = \pi\log(2) + 4\,\frac{\partial}{\partial\epsilon}\int_0^{\pi/2}\cos(t)^\epsilon\,dt\bigg|_{\epsilon=0}$$
$$= \pi\log(2) + 2\,\frac{\partial}{\partial\epsilon}\,\frac{\Gamma\left(\frac{1+\epsilon}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\epsilon}{2}+1\right)}\Bigg|_{\epsilon=0} = \pi\log(2) + 2\,\left(-\pi\log(2)\right) = -\pi\log(2)$$

(865.61) $\displaystyle\int_0^\infty \log\left(1+\frac{a^2}{z^2}\right)\cos\left(m\,z\right)\,dz$

$$\int_0^\infty \log\left(1+\frac{a^2}{z^2}\right)\cos\left(m\,z\right)\,dz = \frac{\sqrt{\pi}\,a}{2}\,\mathrm{G}\left(\frac{1}{2};;0,\frac{1}{2};-\frac{1}{2};\frac{a^2\,m^2}{4}\right)$$
$$= \frac{\sqrt{\pi}\,a}{2}\,\mathrm{G}\left(0,\frac{1}{2};;0,\frac{1}{2};-\frac{1}{2},0;\frac{a^2\,m^2}{4}\right) = \frac{\sqrt{\pi}\,a}{2}\cdot 2\sqrt{\pi}\,\mathrm{G}\left(0;;0;-1;a\,m\right)$$
$$= \pi\,a\,\mathrm{G}\left(0;;0;-1;a\,m\right) = \pi\,a\cdot\frac{\Gamma(1)}{\Gamma(2)}\,{}_1\mathrm{F}_1\left(1;2;-a\,m\right)$$
$$= \pi\,a\cdot\frac{1}{(-a\,m)}\,\left(e^{-a\,m}-1\right) = \frac{\pi}{m}\,\left(1-e^{-a\,m}\right)$$

(865.62) $\displaystyle\int_0^\infty \log\left(\frac{a^2+z^2}{b^2+z^2}\right)\cos\left(m\,z\right)\,dz$

$$\int_0^\infty \log\left(\frac{a^2+z^2}{b^2+z^2}\right)\cos\left(m\,z\right)\,dz$$
$$= \int_0^\infty \log\left(1+\frac{a^2}{z^2}\right)\cos\left(m\,z\right)\,dz - \int_0^\infty \log\left(1+\frac{b^2}{z^2}\right)\cos\left(m\,z\right)\,dz$$
$$= \frac{\pi}{m}\,\left(1-e^{-a\,m}\right) - \frac{\pi}{m}\,\left(1-e^{-b\,m}\right) = \frac{\pi}{m}\,\left(e^{-b\,m}-e^{-a\,m}\right)$$

(865.73(2)) $\displaystyle\int_0^{2\pi} \log\left(a^2 - 2\,a\,b\cos(z) + b^2\right)\,dz$

$$\mathrm{I}(t) = \int_0^{2\pi} \log\left(t^2 - 2\,t\,b\cos(z) + b^2\right)\,dz$$

$t > b > 0$

$$\frac{\partial}{\partial t}\,\mathrm{I}(t) = \int_0^{2\pi} \frac{2\,t - 2\,b\cos(z)}{t^2 - 2\,t\,b\cos(z) + b^2}\,dz = \frac{4\,\pi}{t}$$

$$\mathrm{I}(a) - \mathrm{I}(b) = \int_b^a \frac{4\,\pi}{t}\,dt = 4\,\pi\log\left(\frac{a}{b}\right)$$

$$\mathrm{I}(b) = \int_0^{2\pi} \log\left(2\,b^2 - 2\,b^2\cos(z)\right)\,dz = 2\,\pi\log\left(2\,b^2\right) + \int_0^{2\pi} \log\left(1 - \cos(z)\right)\,dz$$
$$= 2\,\pi\log\left(2\,b^2\right) - 2\,\pi\log(2) = 4\,\pi\log(b)$$

$$\mathrm{I}(a) = 4\,\pi\log\left(\frac{a}{b}\right) - 4\,\pi\log(b) = 4\,\pi\log(a)$$

$a > b > 0$

(865.903) $\displaystyle\int_0^{\infty} z^2\,e^{-a\,z}\log\left(\frac{1}{z}\right)\,dz$

$$\int_0^{\infty} z^2\,e^{-a\,z}\log\left(\frac{1}{z}\right)\,dz = -\frac{\partial}{\partial\epsilon}\int_0^{\infty} z^{2+\epsilon}\,e^{-a\,z}\,dz\bigg|_{\epsilon=0} = -\frac{\partial}{\partial\epsilon}\,a^{-3-\epsilon}\,\Gamma\left(3+\epsilon\right)\bigg|_{\epsilon=0}$$
$$= a^{-3-\epsilon}\,\Gamma\left(3+\epsilon\right)\,\left(\log(a) - \psi\left(3+\epsilon\right)\right)\bigg|_{\epsilon=0} = \frac{2\log(a) - 3 + 2\,\gamma}{a^3}$$

(866.01(1)) $\displaystyle\int_0^{\pi} \cos\left(a\cos(z)\right)\,dz$

$$\int_0^{\pi} \cos\left(a\cos(z)\right)\,dz = 2\int_0^{\pi/2} \cos\left(a\cos(z)\right)\,dz = 2\int_0^{1} \frac{\cos\left(a\,t\right)}{\sqrt{1-t^2}}\,dt$$

$$\left(1-t^2\right)^{-1/2}\,\mathrm{H}\left(1-|\,t\,|\right) = \Gamma\left(\frac{1}{2}\right)\,\mathrm{G}\left(;\frac{1}{2};0;;t^2\right)$$

$$\cos\left(a\,t\right) = \sqrt{\pi}\,\mathrm{G}\left(;;0;\frac{1}{2};\frac{a^2\,t^2}{4}\right)$$

$$\begin{aligned}\int_0^{\pi} \cos\left(a\cos(z)\right)\,dz &= 2\,\pi\int_0^{\infty} \mathrm{G}\left(;\frac{1}{2};0;;t^2\right)\,\mathrm{G}\left(;;0;\frac{1}{2};\frac{a^2\,t^2}{4}\right)\,dt\\ &= \pi\cdot\frac{2}{a}\,\mathrm{G}\left(\frac{1}{2};\frac{1}{2},0;0;;\frac{4}{a^2}\right) = \frac{2\,\pi}{a}\,\mathrm{G}\left(\frac{1}{2};\frac{1}{2};;;\frac{4}{a^2}\right) = \frac{2\,\pi}{a}\,\mathrm{G}\left(;;\frac{1}{2};\frac{1}{2};\frac{a^2}{4}\right)\\ &= \pi\,\mathrm{G}\left(;;0;0;\frac{a^2}{4}\right) = \pi\,\mathrm{J}_0(a)\end{aligned}$$

Reference:

Dwight, Herbert B. (1961), *Tables of Integrals and Other Mathematical Data, Fourth Edition*, MacMillan Publishing Co., Inc., New York.